

# Spectral-Galerkin Approximation and Optimal Error Estimate for Stokes Eigenvalue Problems in Polar Geometries\*

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## Abstract

In this paper we propose and analyze spectral-Galerkin methods for the Stokes eigenvalue problem based on the stream function formulation in polar geometries. We first analyze the stream function formulated fourth-order equation under the polar coordinates, then we derive the pole condition and reduce the problem on a circular disk to a sequence of equivalent one-dimensional eigenvalue problems that can be solved in parallel. The novelty of our approach lies in the construction of suitably weighted Sobolev spaces according to the pole conditions, based on which, the optimal error estimate for approximated eigenvalue of each one dimensional problem can be obtained. Further, we extend our method to the non-separable Stokes eigenvalue problem in an elliptic domain and establish the optimal error bounds. Finally, we provide some numerical experiments to validate our theoretical results and algorithms.

**Keywords:** Stokes eigenvalue problem, polar geometry, pole condition, spectral-Galerkin approximation, optimal error analysis

## 1 Introduction

We consider in this paper the Stokes eigenvalue problem which arises in stability analysis of the stationary solution of the Navier-Stokes equations [20]:

$$-\Delta \mathbf{u} + \nabla p = \lambda \mathbf{u}, \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad (1.2)$$

$$\mathbf{u} = 0, \quad \text{on } \partial\Omega, \quad (1.3)$$

where  $\mathbf{u} = (u_1, u_2)$  is the flow velocity,  $p$  is the pressure,  $\Delta$  is the Laplacian operator,  $\Omega$  is the flow domain and  $\partial\Omega$  denotes the boundary of the flow domain  $\Omega$ .

Let us introduce the stream function  $\psi$  such that  $\mathbf{u} = (\partial_y \psi, -\partial_x \psi)$ . Then we derive an alternative formulation for (1.1)-(1.3):

$$-\Delta^2 \psi = \lambda \Delta \psi, \quad \text{in } \Omega, \quad (1.4)$$

$$\psi = \frac{\partial \psi}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega, \quad (1.5)$$

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where  $\mathbf{n}$  is the unit outward normal to the boundary  $\partial\Omega$ . (1.4) is also referred to as the biharmonic eigenvalue problem for plate buckling. The naturally equivalent weak form of (1.4)-(1.5) reads: Find  $(\lambda, \psi) \in \mathbb{R} \times H_0^2(\Omega)$  such that

$$\mathcal{A}(\psi, \phi) = \lambda \mathcal{B}(\psi, \phi), \quad \phi \in H_0^2(\Omega), \quad (1.6)$$

where the bilinear forms  $\mathcal{A}$  and  $\mathcal{B}$  are defined by

$$\begin{aligned} \mathcal{A}(\psi, \phi) &= (\Delta\psi, \Delta\phi) = \int_{\Omega} \Delta\psi \Delta\bar{\phi} \, dx dy, \\ \mathcal{B}(\psi, \phi) &= (\nabla\psi, \nabla\phi) = \int_{\Omega} \nabla\psi \cdot \nabla\bar{\phi} \, dx dy. \end{aligned}$$

There are various numerical approaches to solving (1.4)-(1.5). Mixed finite element methods introduce the auxiliary function  $w = \Delta\psi$  to reduce the fourth-order equation to a saddle point problem and then discretize the reduced second order equations with ( $C^0$ -) continuous finite elements [8, 22, 10, 29]. However, spurious solutions may occur in some situations. The conforming finite element methods including Argyris elements [2] and the partition of unity finite elements [11], require globally continuously differentiable finite element spaces, which are difficult to construct and implement. The third type of approaches use non-conforming finite element methods, such as Adini elements [1], Morley elements [19, 21, 25] and the ordinary  $C^0$ -interior penalty Galerkin method [26]. Their disadvantage lies in that such elements do not come in a natural hierarchy. Both the conforming and nonconforming finite element methods are based on the naturally equivalent variational formulation (1.6), and usually involve low order polynomials and guarantee only a low order of convergence.

In contrast, it is observed in [31] that the spectral method, whenever it is applicable, has tremendous advantage over the traditional  $h$ -version methods. In particular, spectral and spectral element methods using high order orthogonal polynomials for fourth-order equations result in an exponential order of convergence for smooth solutions [23, 6, 5, 13, 30, 14, 9]. In analogy to the Argyris finite element methods, the conforming spectral element method requires globally continuously differentiable element spaces, which are extremely difficult to construct and implement on unstructured (triangular or quadrilateral) meshes. This is exactly the reason why  $C^1$ -conforming spectral elements are rarely reported in literature except those on rectangular meshes [30]. Hence, the spectral methods using globally smooth basis functions are naturally suitable choices in practice for (1.6) on some fundamental regions including rectangles, triangles and polar geometries.

To the best of our knowledge there are few reports on spectral-Galerkin approximation for the Stokes eigenvalue problem by the stream function formulation in polar geometries. The polar transformation introduces polar singularities and variable coefficients of the form  $r^{\pm m}$  in polar coordinates [23, 4], which involves intricate pole conditions thus brings forth severe difficulties in both the design of approximation schemes and the corresponding error analysis. The aim of the current paper is to propose and analyze an efficient spectral-Galerkin approximation for the stream function formulation of the Stokes eigenvalue problem in polar geometries. As the first step, we use the separation of variables in polar coordinates to reduce the original problem in the unit disk to equivalent infinite sequence of one-dimensional eigenvalue problems which can be solved individually in parallel. Rigorous pole conditions involved are prerequisite for the equivalence of the original problem and the sequence of the one-dimensional eigenvalue problems, and thus play a fundamental role in our further study. It is worthy to note, however, that the pole conditions derived for the fourth-order source problems in open literature (such as [23, 4]) are inadequate for our eigenvalue problems since they would inevitably induce improper/spurious computational results.

Based on the pole condition, suitable approximation spaces are introduced and spectral-Galerkin schemes are proposed. A rigorous analysis on the optimal error estimate in certain properly introduced weighted Sobolev spaces is made for each one dimensional eigenvalue problem by using the minimax principle. Finally, we extend our spectral-Galerkin method to solving the stream function formulation of the Stokes eigenvalue problem in an elliptic region. Owing to its non-separable

property, this problem is actually another challenge both in computation and analysis. A brief explanation on the implementation of the approximation scheme is first given, and an optimal error estimate is then presented in the Cartesian coordinates under the framework of Babuška and Osborn [3].

The rest of this paper is organized as follows. In the next section, dimension reduction scheme of the Stokes eigenvalue problem is presented. In §3, we derive the weak formulation and prove the error estimation for a sequence of equivalent one-dimensional eigenvalue problems. Also, we describe the details for an efficient implementation of the algorithm. In §4, we extend our algorithm to the case of elliptic region. We present several numerical experiments in §5 to demonstrate the accuracy and efficiency of our method. Finally, in §6 we give some concluding remarks.

## 2 Dimensionality reduction and pole conditions

Before coming to the main body of this section, we would like to introduce some notations and conventions which will be used throughout the paper. Let  $\omega$  be a generic positive weight function on a bounded domain  $\Omega$ , which is not necessarily in  $L^1(\Omega)$ . Denote by  $(u, v)_{\omega, \Omega}$  the inner product of  $L^2_\omega(\Omega)$  whose norm is denoted by  $\|\cdot\|_{\omega, \Omega}$ . We use  $H^m_\omega(\Omega)$  and  $H^m_{0, \omega}(\Omega)$  to denote the usual weighted Sobolev spaces, whose norm is denoted by  $\|\cdot\|_{m, \omega, \Omega}$ . In cases where no confusion would arise,  $\omega$  (if  $\omega = 1$ ) and  $\Omega$  may be dropped from the notation. Let  $\mathbb{N}_0$  (resp.  $\mathbb{Z}$ ) be the collection of nonnegative integers (resp. integers). For  $N \in \mathbb{N}_0$ , we denote by  $\mathbb{P}_N(\Omega)$  the collection of all algebraic polynomials on  $\Omega$  with the total degree no greater than  $N$ . We denote by  $c$  a generic positive constant independent of any function and of any discretization parameters. We use the expression  $A \lesssim B$  to mean that  $A \leq cB$ .

In the current section, we restrict our attention to the unit disk  $\Omega = D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ . We shall employ a classical technique, separation of variables, to reduce the problem to a sequence of equivalent one-dimensional problems.

Throughout this paper, we shall use the polar coordinates  $(r, \theta)$  for points in the disk  $D$  such that  $(x, y) = (r \cos \theta, r \sin \theta)$ . We associate any function  $u(x, y)$  in Cartesian coordinates with its partner  $\tilde{u}(r, \theta) = u(r \cos \theta, r \sin \theta)$  in polar coordinates. If no confusion would arise, we shall use the same notation  $u$  for  $u(x, y)$  and  $\tilde{u}(r, \theta)$ . We now recall that, under the polar coordinates,

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}, \quad \nabla = \left( \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right)^t. \quad (2.1)$$

Then the bilinear forms  $\mathcal{A}$  and  $\mathcal{B}$  in (1.6) become

$$\begin{aligned} \mathcal{A}(\psi, \phi) &= \int_0^1 r dr \int_0^{2\pi} \left[ \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right] \left[ \frac{\partial^2 \bar{\phi}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{\phi}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{\phi}}{\partial \theta^2} \right] d\theta, \\ \mathcal{B}(\psi, \phi) &= \int_0^1 r dr \int_0^{2\pi} \left[ \frac{\partial \psi}{\partial r} \frac{\partial \bar{\phi}}{\partial r} + \frac{1}{r^2} \frac{\partial \psi}{\partial \theta} \frac{\partial \bar{\phi}}{\partial \theta} \right] d\theta. \end{aligned}$$

Denote  $I = (0, 1)$  and define the bilinear forms for functions  $u, v$  on  $I$ ,

$$\begin{aligned} \mathcal{A}_m(u, v) &= \int_0^1 \left[ u'' + \frac{u'}{r} - \frac{m^2}{r^2} u \right] \left[ \bar{v}'' + \frac{\bar{v}'}{r} - \frac{m^2}{r^2} \bar{v} \right] r dr, \\ \mathcal{B}_m(u, v) &= \int_0^1 \left( r u' \bar{v}' + \frac{m^2}{r} u \bar{v} \right) dr. \end{aligned}$$

Further let us assume

$$\psi = \sum_{m \in \mathbb{Z}} \psi_m(r) e^{im\theta}, \quad \phi = \sum_{m \in \mathbb{Z}} \phi_m(r) e^{im\theta}. \quad (2.2)$$

By the orthogonality of the Fourier system  $\{e^{im\theta}\}$ , one finds that

$$\mathcal{A}(\psi, \phi) = 2\pi \sum_{m \in \mathbb{Z}} \mathcal{A}_m(\psi_m, \phi_m), \quad \mathcal{B}(\psi, \phi) = 2\pi \sum_{m \in \mathbb{Z}} \mathcal{B}_m(\psi_m, \phi_m).$$

For the well-posedness of  $\mathcal{B}_m(\psi_m, \phi_m)$  and  $\mathcal{A}_m(\psi_m, \phi_m)$ , the following pole conditions for  $\psi_m$  (and the same type of pole conditions for  $\phi_m$ ) should be imposed,

$$m\psi_m(0) = 0, \quad \lim_{r \rightarrow 0^+} \left[ \psi'_m(r) - \frac{m^2}{r} \psi_m(r) \right] = (1 - m^2)\psi'_m(0) = 0, \quad (2.3)$$

which can be further simplified into the following three categories,

$$(1). \quad \psi'_m(0) = 0, \quad m = 0; \quad (2.4)$$

$$(2). \quad \psi_m(0) = 0, \quad |m| = 1; \quad (2.5)$$

$$(3). \quad \psi_m(0) = \psi'_m(0) = 0, \quad |m| \geq 2. \quad (2.6)$$

It is worthy to note that our pole condition (2.5) for  $|m| = 1$  is a revision of the pole condition  $\psi'_m(0) = \psi_m(0) = 0$  in (4.8) of [23]. A concrete example to support the absence of  $\psi'_{\pm 1}(0) = 0$  reads,

$$\psi = \psi_{\pm 1}(r)e^{\pm i\theta} \in H_0^2(D), \quad \psi_{\pm 1}(r) = (1 - r)^2 r.$$

Also, this absence of  $\psi'_{\pm 1}(0) = 0$  in (2.5) is also confirmed by [7].

The boundary conditions  $\psi = \partial_n \psi = 0$  on  $\partial D$  states  $\psi'_m(1) = \psi_m(1) = 0$  for all integer  $m$ . Meanwhile,  $\mathcal{A}_m(\psi_m, \psi_m) = 0$  together with  $\psi'_m(1) = \psi_m(1) = 0$  implies  $\psi_m \equiv 0$ . It is then easy to verify that  $\sqrt{\mathcal{A}_m(\psi_m, \psi_m)}$  (resp.  $\sqrt{\mathcal{B}_m(\psi_m, \psi_m)}$ ) induces a Sobolev norm for any function  $\psi_m$  on  $I$  which satisfies the boundary condition  $\psi'_m(1) = \psi_m(1) = 0$  (resp.  $\psi_m(1) = 0$ ) and the pole condition  $(1 - m^2)\psi'_m(0) = m\psi_m(0) = 0$  (resp.  $m\psi_m(0) = 0$ ).

We now introduce two non-uniformly weighted Sobolev spaces on  $I$ ,

$$\mathring{H}_m^1(I) := \{u : \mathcal{B}_m(u, u) < \infty, \quad mu(0) = u(1) = 0\}, \quad (2.7)$$

$$\mathring{H}_m^2(I) := \{u : \mathcal{A}_m(u, u) < \infty, \quad mu(0) = (1 - m^2)u'(0) = u(1) = u'(1) = 0\}, \quad (2.8)$$

which are endowed with energy norms

$$\|u\|_{1,m,I} = \sqrt{\mathcal{B}_m(u, u)}, \quad \|u\|_{2,m,I} = \sqrt{\mathcal{A}_m(u, u)}. \quad (2.9)$$

In the sequel, (1.6) is reduced to a system of infinite one-dimensional eigen problems: to find  $(\lambda_m, \psi_m) \in \mathbb{R} \times \mathring{H}_m^2(I)$  such that  $\|\psi_m\|_{1,m,I} = 1$  and

$$\mathcal{A}_m(\psi_m, \phi_m) = \lambda_m \mathcal{B}_m(\psi_m, \phi_m), \quad \phi_m \in \mathring{H}_m^2(I), \quad m \in \mathbb{Z}. \quad (2.10)$$

We now conclude this section with the following lemma on  $\mathcal{A}_m(\cdot, \cdot)$  and  $\mathcal{B}_m(\cdot, \cdot)$ .

**Lemma 2.1** For  $u, v \in \mathring{H}_m^2(I)$ ,

$$\mathcal{B}_m(u, v) = \int_0^1 \left( u' \pm \frac{m}{r} u \right) \left( \bar{v}' \pm \frac{m}{r} \bar{v} \right) r dr, \quad (2.11)$$

$$\mathcal{A}_m(u, v) = \int_0^1 \left[ r \left( u' \mp \frac{m}{r} u \right)' \left( \bar{v}' \mp \frac{m}{r} \bar{v} \right) + \frac{(1 \pm m)^2}{r} \left( u' \mp \frac{m}{r} u \right) \left( \bar{v}' \mp \frac{m}{r} \bar{v} \right) \right] dr. \quad (2.12)$$

*Proof.* By integration by parts and the pole condition (2.3), one verifies that

$$\begin{aligned} \int_0^1 \left( u' \pm \frac{m}{r} u \right) \left( \bar{v}' \pm \frac{m}{r} \bar{v} \right) r dr &= \int_0^1 \left( r u' \bar{v}' + \frac{m^2}{r} u \bar{v}' \right) dr \pm m \int_0^1 (u \bar{v})' dr \\ &= \int_0^1 \left( r u' \bar{v}' + \frac{m^2}{r} u \bar{v}' \right) dr, \end{aligned}$$

which gives (2.11).

Next, one readily checks that

$$\begin{aligned} u'' + \frac{u'}{r} - \frac{m^2 u}{r^2} &= \left(u' \mp \frac{m}{r} u\right)' + \frac{1 \pm m}{r} \left(u' \mp \frac{m}{r} u\right), \\ v'' + \frac{v'}{r} - \frac{m^2 v}{r^2} &= \left(v' \mp \frac{m}{r} v\right)' + \frac{1 \pm m}{r} \left(v' \mp \frac{m}{r} v\right). \end{aligned}$$

As a result,

$$\begin{aligned} \mathcal{A}_m(u, v) &= \int_0^1 \left[ \left(u' \mp \frac{m}{r} u\right)' + \frac{1 \pm m}{r} \left(u' \mp \frac{m}{r} u\right) \right] \left[ \left(v' \mp \frac{m}{r} v\right)' + \frac{1 \pm m}{r} \left(v' \mp \frac{m}{r} v\right) \right] r dr \\ &= \int_0^1 \left[ r \left(u' \mp \frac{m}{r} u\right)' \left(v' \mp \frac{m}{r} v\right)' + \frac{(1 \pm m)^2}{r} \left(u' \mp \frac{m}{r} u\right) \left(v' \mp \frac{m}{r} v\right) \right] dr \\ &\quad + \int_0^1 (1 \pm m) \left[ \left(u' \mp \frac{m}{r} u\right) \left(v' \mp \frac{m}{r} v\right) \right]' dr. \end{aligned}$$

Meanwhile, the pole conditions (2.4)-(2.6) states that both  $(1 \pm m)(u' \mp \frac{m}{r} u)$  and  $(1 \pm m)(v' \mp \frac{m}{r} v)$  vanish at the two endpoints of  $I$ . Thus the last integral above is zero, and (2.12) is now proved. ■

### 3 Spectral Galerkin approximation and its error estimates

Let  $P_N(I)$  be the space of polynomials of degree less than or equal to  $N$  on  $I$ , and setting  $X_N^m = \mathbb{P}_N(I) \cap \mathring{H}_m^2(I)$ . Then the spectral Galerkin approximation scheme to (2.10) is: Find  $(\lambda_{mN}, \psi_{mN}) \in \mathbb{R} \times X_N^m$  such that  $\|\psi_{mN}\|_{1,m} = 1$  and

$$\mathcal{A}_m(\psi_{mN}, v_N) = \lambda_{mN} \mathcal{B}_m(\psi_{mN}, v_N), \quad \forall v_N \in X_N^m. \quad (3.1)$$

Due to the symmetry properties  $\mathcal{A}_m = \mathcal{A}_{-m}$  and  $\mathcal{B}_m = \mathcal{B}_{-m}$ , we shall only consider  $m \in \mathbb{N}_0$  from now on in this section.

#### 3.1 Mini-max principle

To give the error analysis, we will use extensively the minimax principle.

**Lemma 3.1** *Let  $\lambda_m^l$  denote the eigenvalues of (2.10) and  $V_l$  be any  $l$ -dimensional subspace of  $\mathring{H}_m^2(I)$ . Then, for  $\lambda_m^1 \leq \lambda_m^2 \leq \dots \leq \lambda_m^l \leq \dots$ , there holds*

$$\lambda_m^l = \min_{V_l \subset \mathring{H}_m^2(I)} \max_{v \in V_l} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(v, v)}. \quad (3.2)$$

*Proof.* See Theorem 3.1 in [18]. ■

**Lemma 3.2** *Let  $\lambda_m^i$  denote the eigenvalues of (2.10) and be arranged in an ascending order, and define*

$$E_{i,j} = \text{span} \{ \psi_m^i, \dots, \psi_m^j \},$$

where  $\psi_m^i$  is the eigenfunction corresponding to the eigenvalue  $\lambda_m^i$ . Then we have

$$\lambda_m^l = \max_{v \in E_{k,l}} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(v, v)} \quad k \leq l, \quad (3.3)$$

$$\lambda_m^l = \min_{v \in E_{l,m}} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(v, v)} \quad l \leq m. \quad (3.4)$$

*Proof.* See Lemma 3.2 in [18]. ■

It is true that the minimax principle is also valid for the discrete formulation (3.1) (see [18]).

**Lemma 3.3** *Let  $\lambda_{mN}^l$  denote the eigenvalues of (3.1), and  $V_l$  be any  $l$ -dimensional subspace of  $X_N^m$ . Then, for  $\lambda_{mN}^1 \leq \lambda_{mN}^2 \leq \dots \leq \lambda_{mN}^l \leq \dots$ , there holds*

$$\lambda_{mN}^l = \min_{V_l \subset X_N^m} \max_{v \in V_l} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(v, v)}. \quad (3.5)$$

Define the orthogonal projection  $\Pi_N^{2,m} : \mathring{H}_m^2(I) \mapsto X_N^m$  such that

$$\mathcal{A}_m(\psi_m - \Pi_N^{2,m} \psi_m, v) = 0, \quad \forall v \in X_N^m. \quad (3.6)$$

**Theorem 3.1** *Let  $\lambda_{mN}^l$  be obtained by solving (3.1) as an approximation of  $\lambda_m^l$ , an eigenvalue of (2.10). Then, we have*

$$0 < \lambda_m^l \leq \lambda_{mN}^l \leq \lambda_m^l \max_{v \in E_{1,l}} \frac{\mathcal{B}_m(v, v)}{\mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)}. \quad (3.7)$$

*Proof.* According to the coerciveness of  $\mathcal{A}_m(u, v)$  and  $\mathcal{B}_m(u, v)$  we easily derive  $\lambda_m^l > 0$ . Since  $X_N^m \subset \mathring{H}_m^2(I)$ , from (3.2) and (3.5) we can obtain  $\lambda_m^l \leq \lambda_{mN}^l$ . Let  $\Pi_N^{2,m} E_{1,l}$  denote the space spanned by  $\Pi_N^{2,m} \psi_m^1, \Pi_N^{2,m} \psi_m^2, \dots, \Pi_N^{2,m} \psi_m^l$ . It is obvious that  $\Pi_N^{2,m} E_{1,l}$  is a  $l$ -dimensional subspace of  $X_N^m$ . From the minimax principle, we have

$$\lambda_{mN}^l \leq \max_{v \in \Pi_N^{2,m} E_{1,l}} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(v, v)} = \max_{v \in E_{1,l}} \frac{\mathcal{A}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)}{\mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)}.$$

Since  $\mathcal{A}_m(v, v) = \mathcal{A}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v) + 2a_m(v - \Pi_N^{2,m} v, \Pi_N^{2,m} v) + \mathcal{A}_m(v - \Pi_N^{2,m} v, v - \Pi_N^{2,m} v)$ , from  $\mathcal{A}_m(v - \Pi_N^{2,m} v, \Pi_N^{2,m} v) = 0$  and the non-negativity of  $a(v - \Pi_N^{2,m} v, v - \Pi_N^{2,m} v)$ , we have

$$\mathcal{A}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v) \leq \mathcal{A}_m(v, v).$$

Thus, we have

$$\begin{aligned} \lambda_{mN}^l &\leq \max_{v \in E_{1,l}} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)} \\ &= \max_{v \in E_{1,l}} \frac{\mathcal{A}_m(v, v)}{\mathcal{B}_m(v, v)} \frac{\mathcal{B}_m(v, v)}{\mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)} \\ &\leq \lambda_m^l \max_{v \in E_{1,l}} \frac{\mathcal{B}_m(v, v)}{\mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)}. \end{aligned}$$

The proof of Theorem 3.1 is completed. ■

### 3.2 Error estimates

Denote by  $\omega^{\alpha, \beta} := \omega^{\alpha, \beta}(r) = (1-r)^{\alpha} r^{\beta}$  the Jacobi weight function of index  $(\alpha, \beta)$ , which is not necessarily in  $L^1(I)$ . Define the  $L^2$ -orthogonal projection  $\pi_N^{0,0} : L^2(I) \mapsto \mathbb{P}_N(I)$  such that

$$(\pi_N^{0,0} u - u, v)_I = 0, \quad v \in \mathbb{P}_N(I).$$

Further, for  $k \geq 1$ , define recursively the  $H^k$ -orthogonal projections  $\pi_N^{-k,-k} : H^k(I) \mapsto \mathbb{P}_N(I)$  such that

$$[\pi_N^{-k,-k}u](r) = \int_0^r [\pi_{N-1}^{1-k,1-k}u'](t)dt + u(0).$$

Next, for any nonnegative integers  $s \geq k \geq 0$ , define the Sobolev space

$$H^{s,k}(I) = \left\{ u \in H^k(I) : \sum_{l=0}^s \|\partial_r^l u\|_{\omega^{\max(l-k,0), \max(l-k,0), I}} < \infty \right\}.$$

Now we have the following error estimate on  $\pi_N^{-k,-k}$ .

**Lemma 3.4** ([15, Theorem 3.1.4])  $\pi_N^{-k,-k}u$  is a Legendre tau approximation of  $u$  such that

$$\partial_r^l [\pi_N^{-k,-k}u](0) = \partial_r^l u(0), \quad \partial_r^l [\pi_N^{-k,-k}u](1) = \partial_r^l u(1), \quad 0 \leq l \leq k-1, \quad (3.8)$$

$$(\pi_N^{-k,-k}u - u, v) = 0, \quad v \in \mathbb{P}_{N-2k}. \quad (3.9)$$

Further suppose  $u \in H^{s,k}(I)$  with  $s \geq k$ . Then for  $N \geq k$ ,

$$\|\partial_r^l (\pi_N^{-k,-k}u - u)\|_{\omega^{l-k, l-k, I}} \lesssim N^{l-s} \|\partial_r^s u\|_{\omega^{s-k, s-k, I}}, \quad 0 \leq l \leq k \leq s. \quad (3.10)$$

**Theorem 3.2** Suppose  $u \in \dot{H}_m^2(I)$  and  $u' + \frac{m}{r}u \in H^{s-1,1}(I)$  with  $s \geq 2$  and  $m \in \mathbb{N}_0$ . Then for  $N \geq m+3$ ,

$$\|\Pi_N^{2,m}u - u\|_{2,m,I} \lesssim (N+m)N^{1-s} \left\| \partial_r^{s-1} \left( \partial_r + \frac{m}{r} \right) u \right\|_{\omega^{s-2, s-2, I}}. \quad (3.11)$$

*Proof.* Define the differential operator  $\mathcal{D}_m = \partial_r + \frac{m}{r} = \frac{1}{r^m} \partial_r (r^m \cdot)$  and then set

$$u_N(r) = -\frac{1}{r^m} \int_r^1 t^m [\pi_{N-1}^{-1,-1} \mathcal{D}_m u](t) dt.$$

We shall first prove  $u_N \in X_N^m$ . By (3.9), we find that

$$\int_0^1 t^m [\pi_{N-1}^{-1,-1} \mathcal{D}_m u](t) dt = \int_0^1 t^m [\mathcal{D}_m u](t) dt = \int_0^1 \partial_t [t^m u(t)] dt = 0, \quad N \geq m+3, m \neq 0,$$

where the last equality sign is derived from the boundary condition  $u(1) = 0$ . Moreover,

$$\partial_r \int_r^1 t^m [\pi_{N-1}^{-1,-1} \mathcal{D}_m u](t) dt = -r^m [\pi_{N-1}^{-1,-1} \mathcal{D}_m u](r).$$

As a result,  $u_N \in \mathbb{P}_N(I)$  and

$$u_N(0) = -\lim_{r \rightarrow 0} \frac{1}{r^m} \int_r^1 t^m [\pi_{N-1}^{-1,-1} \mathcal{D}_m u](t) dt = 0, \quad m \neq 0.$$

Further,  $u \in \dot{H}_m^2(I)$  implies

$$[\mathcal{D}_m u](1) = 0, \quad m \in \mathbb{Z}; \quad [\mathcal{D}_m u](0) = 0, \quad m \neq 1,$$

which, together with the property (3.8) of  $\pi_N^{-1,-1}$ , gives

$$[\pi_{N-1}^{-1,-1} \mathcal{D}_m u](1) = 0, \quad m \in \mathbb{Z}; \quad [\pi_{N-1}^{-1,-1} \mathcal{D}_m u](0) = 0, \quad m \neq 1.$$

In the sequel, we deduce that  $u'_N(0) = 0$  if  $m \neq 1$  and  $u_N(1) = u'_N(1) = 0$ . In summary, we conclude that  $u_N \in X_N^m$ .

Next by (2.12) and (3.10), we have

$$\begin{aligned}\|u_N - u\|_{2,m}^2 &= \|\partial_r(\pi_{N-1}^{-1,-1} - I)\mathcal{D}_m u\|_{\omega^{0,1}}^2 + (m-1)^2 \|(\pi_{N-1}^{-1,-1} - I)\mathcal{D}_m u\|_{\omega^{0,-1}}^2 \\ &\leq \|\partial_r(\pi_{N-1}^{-1,-1} - I)\mathcal{D}_m u\|_{\omega^{0,0}}^2 + (m-1)^2 \|(\pi_{N-1}^{-1,-1} - I)\mathcal{D}_m u\|_{\omega^{-1,-1}}^2 \\ &\lesssim [N^{4-2s} + (m-1)^2 N^{2-2s}] \|\partial_r^{s-1} \mathcal{D}_m u\|_{\omega^{s-2,s-2}}^2.\end{aligned}$$

Finally, (3.11) is an immediate consequence of the projection theorem,

$$\|\Pi_N^{2,m} u - u\|_{2,m,I} = \inf_{v \in X_N^m} \|v - u\|_{2,m,I} \leq \|u_N - u\|_{2,m,I}.$$

The proof is now completed. ■

**Theorem 3.3** *Let  $\lambda_{mN}^l$  is the  $l$ -th approximate eigenvalue of  $\lambda_m^l$ . If  $\{\psi_m^i\}_{i=1}^l \subset \dot{H}_m^2(I) \cap H^{s,2}(I)$  with  $s \geq 2$ , then we have*

$$|\lambda_{mN}^l - \lambda_m^l| \lesssim (N^2 + m^2) N^{2-2s} \max_{1 \leq i \leq l} \left\| \partial_r^{s-1} \left( \partial_r + \frac{m}{r} \right) \psi_m^i \right\|_{\omega^{s-2,s-2},I}^2.$$

*Proof.* For any  $0 \neq v \in E_{1,l}$ , it can be represented by  $v = \sum_{i=1}^l \mu_i \psi_m^i$ ; we then have

$$\begin{aligned}\frac{\mathcal{B}_m(v, v) - \mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)}{\mathcal{B}_m(v, v)} &\leq \frac{2|\mathcal{B}_m(v, v - \Pi_N^{2,m} v)|}{\mathcal{B}_m(v, v)} \\ &\leq \frac{2 \sum_{i,j=1}^l |\mu_i| |\mu_j| |\mathcal{B}_m(\psi_m^i - \Pi_N^{2,m} \psi_m^i, \psi_m^j)|}{\sum_{i=1}^l |\mu_i|^2} \\ &\leq 2l \max_{i,j=1,\dots,l} |\mathcal{B}_m(\psi_m^i - \Pi_N^{2,m} \psi_m^i, \psi_m^j)| := \varepsilon.\end{aligned}$$

Meanwhile, by the variational form (2.10), the definition (3.6) of  $\Pi_N^{2,m}$ , Cauchy-Schwarz inequality and Theorem 3.2, we have

$$\begin{aligned}|\mathcal{B}_m(\psi_m^i - \Pi_N^{2,m} \psi_m^i, \psi_m^j)| &= \frac{1}{\lambda_m^j} |\lambda_m^j b_m(\psi_m^j, \psi_m^i - \Pi_N^{2,m} \psi_m^i)| \\ &= \frac{1}{\lambda_m^j} |\mathcal{A}_m(\psi_m^j, \psi_m^i - \Pi_N^{2,m} \psi_m^i)| = \frac{1}{\lambda_m^j} |\mathcal{A}_m(\psi_m^j - \Pi_N^{2,m} \psi_m^j, \psi_m^i - \Pi_N^{2,m} \psi_m^i)| \\ &\leq \frac{1}{\lambda_m^j} \|\psi_m^j - \Pi_N^{2,m} \psi_m^j\|_{2,m,I} \|\psi_m^i - \Pi_N^{2,m} \psi_m^i\|_{2,m,I} \\ &\lesssim (N^2 + m^4) N^{2-2s} \|\partial_r^s \psi_m^j\|_{\omega^{s-2,s-2},I} \|\partial_r^s \psi_m^i\|_{\omega^{s-2,s-2},I}.\end{aligned}$$

As a result, we have the following estimate for  $\varepsilon$ ,

$$\varepsilon \lesssim (N^2 + m^2) N^{2-2s} \max_{1 \leq i \leq l} \left\| \partial_r^{s-1} \left( \partial_r + \frac{m}{r} \right) \psi_m^i \right\|_{\omega^{s-2,s-2},I}^2.$$

For sufficiently large  $N$ ,  $\varepsilon < \frac{1}{2}$ . Thus

$$0 < \frac{\mathcal{B}_m(v, v)}{\mathcal{B}_m(\Pi_N^{2,m} v, \Pi_N^{2,m} v)} \leq \frac{1}{1 - \varepsilon} \leq 1 + 2\varepsilon,$$

and we finally deduce from Theorem 3.1 that

$$0 < \lambda_{mN}^l - \lambda_m^l \leq 2\lambda_m^l \varepsilon \lesssim (N^2 + m^2) N^{2-2s} \max_{1 \leq i \leq l} \left\| \partial_r^{s-1} \left( \partial_r + \frac{m}{r} \right) \psi_m^i \right\|_{\omega^{s-2,s-2},I}^2.$$

The proof is now completed. ■



### 3.3 Implementations

We describe in this section how to solve the problems (3.1) efficiently. To this end, we first construct a set of basis functions for  $X_N^m$ . Let

$$\phi_i(r) = (1-r)^2 r^2 J_{i-4}^{2,1}(2r-1), \quad i \geq 4, \quad (3.12)$$

where  $J_k^{\alpha,\beta}$  is the Jacobi polynomial of degree  $k$ .

It is clear that

$$\begin{aligned} X_N^m &= \text{span}\{\phi_i^m = \phi_i : 4 \leq i \leq N\}, \quad m \geq 2 \\ X_N^0 &= \text{span}\{\phi_i^0 = \phi_i : 4 \leq i \leq N\} \oplus \text{span}\{\phi_3^0(r) = \frac{1}{4}(1-r)^2(2r+1)\}, \\ X_N^1 &= \text{span}\{\phi_i^1 = \phi_i : 4 \leq i \leq N\} \oplus \text{span}\{\phi_3^1(r) = \frac{1}{2}(1-r)^2 r\}. \end{aligned}$$

Define  $N_m = 4$  if  $m \geq 2$  and  $N_m = 3$  otherwise. Our basis functions lead to the penta-diagonal matrix  $A^m = [\mathcal{A}_m(\phi_i^m, \phi_j^m)]_{N_m \leq i, j \leq N}$  and the deca-diagonal mass matrix  $B^m = [\mathcal{B}_m(\phi_i^m, \phi_j^m)]_{N_m \leq i, j \leq N}$  instead of the hepta- and hendecagon-diagonal ones in [23].

**Lemma 3.5** For  $i \geq 4$ ,

$$\phi_i''(r) = \frac{(i-3)(i-1)i}{2(2i-3)} J_{i-2}^{0,1}(2r-1) + \frac{2(i-3)^2(i-2)(i-1)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(2r-1) + \frac{(i-4)(i-3)^2}{2(2i-5)} J_{i-4}^{0,1}(2r-1), \quad (3.13)$$

$$\phi_i'(r) = r \left[ \frac{(i-3)i}{2(2i-3)} J_{i-2}^{0,1}(2r-1) - \frac{3(i-3)(i-2)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(2r-1) - \frac{(i-4)(i-3)}{2(2i-5)} J_{i-4}^{0,1}(2r-1) \right] \quad (3.14)$$

$$\begin{aligned} &= \frac{(i-3)i^2}{4(2i-3)(2i-1)} J_{i-1}^{0,1}(2r-1) + \frac{(i-1)(i-3)(2i^2-7i+2)}{2(2i-5)(2i-1)(2i-3)} J_{i-2}^{0,1}(2r-1) - \frac{(i-2)(i-3)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(2r-1) \\ &\quad - \frac{(2i^2-9i+6)(i-3)^2}{2(2i-7)(2i-5)(2i-3)} J_{i-4}^{0,1}(2r-1) - \frac{(i-3)(i-4)^2}{4(2i-5)(2i-7)} J_{i-5}^{0,1}(2r-1), \end{aligned} \quad (3.15)$$

$$\frac{\phi_i(r)}{r} = r \left[ \frac{i-3}{2(2i-3)} J_{i-2}^{0,1}(2r-1) - \frac{2(i-3)(i-2)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(2r-1) + \frac{i-3}{2(2i-5)} J_{i-4}^{0,1}(2r-1) \right] \quad (3.16)$$

$$\begin{aligned} &= \frac{i(i-3)}{4(2i-1)(2i-3)} J_{i-1}^{0,1}(2r-1) - \frac{(i-1)(i-3)}{(2i-1)(2i-3)(2i-5)} J_{i-2}^{0,1}(2r-1) - \frac{(i-2)(i-3)}{2(2i-3)(2i-5)} J_{i-3}^{0,1}(2r-1) \\ &\quad + \frac{(i-3)^2}{2(2i-7)(2i-3)(2i-5)} J_{i-4}^{0,1}(2r-1) + \frac{(i-3)(i-4)}{4(2i-5)(2i-7)} J_{i-5}^{0,1}(2r-1), \end{aligned} \quad (3.17)$$

and

$$\phi_3^{0''}(r) = J_1^{0,1}(2r-1) + \frac{1}{2} J_0^{0,1}(2r-1), \quad \phi_3^{1''}(r) = J_1^{0,1}(2r-1), \quad (3.18)$$

$$\begin{aligned} \phi_3^{0'}(r) &= \frac{r}{2} [J_1^{0,1}(2r-1) - J_0^{0,1}(2r-1)] \\ &= \frac{3}{20} J_2^{0,1}(2r-1) + \frac{1}{10} J_1^{0,1}(2r-1) - \frac{1}{4} J_0^{0,1}(2r-1), \end{aligned} \quad (3.19)$$

$$\begin{aligned} \phi_3^{1'}(r) - \frac{\phi_3^1(r)}{r} &= \frac{r}{3} [J_1^{0,1}(2r-1) - J_0^{0,1}(2r-1)] \\ &= \frac{1}{10} J_2^{0,1}(2r-1) + \frac{1}{15} J_1^{0,1}(2r-1) - \frac{1}{6} J_0^{0,1}(2r-1). \end{aligned} \quad (3.20)$$

Thus for  $j \geq i$ ,

$$A_{i,j}^m = \begin{cases} \frac{(i-3)(i-2)(3i^4+2i^2m^2+3m^4-24i^3-8im^2+70i^2-88i+42)}{4(2i-5)(i-1)(2i-3)} + (\frac{9}{16}\delta_{m,0} + \frac{1}{2}\delta_{m,1})\delta_{i,3}, & j = i, \\ \frac{(i-2)(i-3)(4i^4-4m^4-24i^3+50i^2+10m^2-42i+9)}{4(2i-5)(2i-1)(2i-3)} + (\frac{3}{20}\delta_{m,0} + \frac{1}{4}\delta_{m,1})\delta_{i,3}, & j = i+1, \\ \frac{(i-3)(i-2+m)(i-2-m)(i-m)(i+m)}{8(2i-1)(2i-3)} + \frac{3}{40}\delta_{m,0}\delta_{i,3}, & j = i+2, \\ 0, & j \geq i+3, \end{cases} \quad (3.21)$$

and

$$B_{i,j}^m = \begin{cases} \frac{(i-3)^2(i-2)(5i^2+3m^2-20i+8)}{16(2i-7)(2i-1)(2i-3)(2i-5)} + (\frac{3}{80}\delta_{m,0} + \frac{1}{60}\delta_{m,1})\delta_{i,3}, & j = i, \\ \frac{(i-2)(i-3)(4i^4-24i^3+43i^2+6m^2-21i-26)}{16(2i-7)(2i-5)(2i+1)(2i-1)(2i-3)} + (\frac{1}{140}\delta_{m,0} + \frac{1}{84}\delta_{m,1})\delta_{i,3}, & j = i+1, \\ -\frac{(i-1)^2(i-3)(i^2+m^2-2i-4)}{8(2i-5)(2i+1)(2i-1)(2i-3)} - \frac{3}{560}\delta_{m,0}\delta_{i,3}, & j = i+2, \\ -\frac{i(i-3)(4i^4-8i^3-13i^2+6m^2+17i-6)}{16(2i-5)(2i-3)(2i-1)(2i+3)(2i+1)} - (\frac{3}{280}\delta_{m,0} + \frac{1}{120}\delta_{m,1})\delta_{i,3}, & j = i+3, \\ -\frac{(i-3)i(i+1)(i-m)(i+m)}{32(2i-3)(2i-1)(2i+3)(2i+1)} - (\frac{1}{280}\delta_{m,0} + \frac{1}{315}\delta_{m,1})\delta_{i,3}, & j = i+4, \\ 0, & j \geq i+5. \end{cases} \quad (3.22)$$

We postpone the proof to Appendix B.

We shall look for

$$\psi_{mN} = \sum_{i=N_m}^N \hat{u}_i^m \phi_i^m. \quad (3.23)$$

Now, plugging the expression of (3.23) in (3.1), and taking  $v_N$  through all the basis functions in  $X_N^m$ , we will arrive at the following algebraic linear eigenvalue system:

$$A^m \hat{u}^m = \lambda_N^m B^m \hat{u}^m, \quad (3.24)$$

with

$$A^m = (A_{ij}^m)_{N_m \leq i,j \leq N}, \quad B^m = (B_{ij}^m)_{N_m \leq i,j \leq N}, \quad \hat{u}^m = (\hat{u}_{N_m}^m, \dots, \hat{u}_N^m)^t, \quad (3.25)$$

which can be efficiently solved.

## 4 Extension to elliptic domain

In the section, we extend our algorithm and numerical analysis from a circular disk to an elliptic domain,

$$\Omega = \left\{ (x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\},$$

where  $a$  and  $b$  are the semi-major axis and the semi-minor axis, respectively, i.e.,  $a \geq b > 0$ .

### 4.1 Pole conditions

Let us make the polar transformation  $(x, y) = (ar \cos \theta, br \sin \theta)$ , which maps the rectangle  $R = \{(r, \theta) : 0 \leq r < 1, 0 \leq \theta < 2\pi\}$  in polar coordinates onto the ellipse  $\Omega$  in Cartesian coordinates. For  $s \geq 0$ , we denote  $\mathring{H}^s(R) = \{\tilde{u}(r, \theta) = u(ar \cos \theta, br \sin \theta) : u \in H_0^s(\Omega)\}$ , which is equipped with the norm  $\|u\|_s$ . If no confusion would arise, we shall also use the notation  $u$  for its correspondence  $\tilde{u}$  on  $R$ .

We now revisit the gradient and Laplacian in Cartesian coordinates. It is readily checked that

$$\nabla u = \left( \frac{1}{a} (\cos \theta \partial_r u - \frac{1}{r} \sin \theta \partial_\theta u), \frac{1}{b} (\sin \theta \partial_r u + \frac{1}{r} \cos \theta \partial_\theta u) \right)^t, \quad (4.1)$$

$$\begin{aligned} \Delta u = & \frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \left( \partial_r^2 u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_\theta^2 u \right) \\ & + \frac{1}{2} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \left[ \cos 2\theta \left( \partial_r^2 u - \frac{1}{r} \partial_r u - \frac{1}{r^2} \partial_\theta^2 u \right) + \frac{2 \sin 2\theta}{r} \left( \frac{1}{r} \partial_\theta u - \partial_r \partial_\theta u \right) \right]. \end{aligned} \quad (4.2)$$

Specifically,

$$\begin{aligned} \nabla[u_m(r)e^{im\theta}] &= \left( \frac{1}{2a} \left( u'_m - \frac{m}{r} u_m \right) e^{i(m+1)\theta} + \frac{1}{2a} \left( u'_m + \frac{m}{r} u_m \right) e^{i(m-1)\theta} \right. \\ &\quad \left. - \frac{i}{2b} \left( u'_m - \frac{m}{r} u_m \right) e^{i(m+1)\theta} + \frac{i}{2b} \left( u'_m + \frac{m}{r} u_m \right) e^{i(m-1)\theta} \right)^t, \end{aligned} \quad (4.3)$$

$$\begin{aligned} \Delta[u_m(r)e^{im\theta}] &= \frac{1}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \mathcal{L}_m u_m(r) e^{im\theta} \\ &\quad + \frac{1}{4} \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \left[ \mathcal{K}_m u_m(r) e^{i(m+2)\theta} + \mathcal{K}_{-m} u_m(r) e^{i(m-2)\theta} \right], \end{aligned} \quad (4.4)$$

where  $\mathcal{L}_m$  and  $\mathcal{K}_m$  are differential operators defined by

$$\mathcal{L}_m = \partial_r^2 + \frac{1}{r} \left( \partial_r - \frac{m^2}{r} \right), \quad \mathcal{K}_m = \partial_r^2 - \frac{1+2m}{r} \partial_r + \frac{m^2+2m}{r^2} = \mathcal{L}_m - \frac{2m}{r} \left( \partial_r - \frac{1}{r} \right).$$

To make the  $\nabla[u_m(r)e^{im\theta}]$  and  $\Delta[u_m(r)e^{im\theta}]$  meaningful at the origin, one requires that

$$m u_m(0) = 0, \quad \lim_{r \rightarrow 0^+} \left( u'_m(r) - \frac{m^2 u_m(r)}{r} \right) = 0, \quad m \lim_{r \rightarrow 0^+} \left( u'_m(r) - \frac{u_m(r)}{r} \right) = 0 \quad (4.5)$$

which, as before, can be further simplified into the following three categories,

$$(1). \quad u'_m(0) = 0, \quad m = 0; \quad (4.6)$$

$$(2). \quad u_m(0) = 0, \quad |m| = 1; \quad (4.7)$$

$$(3). \quad u_m(0) = u'_m(0) = 0, \quad |m| \geq 2. \quad (4.8)$$

## 4.2 Spectral-Galerkin approximation and implementation

Define the approximation spaces,

$$X_N = X_{N/2,N}, \quad X_{M,N} = \text{span}\{u_m(r)e^{im\theta} : u_m \in X_N^m, -M \leq m \leq M\}.$$

Then the spectral-Galerkin approximation to (1.6) reads: Find  $(\lambda_N, \psi_N) \in \mathbb{R} \times X_N$  such that  $\|\nabla u\| = 1$  and

$$(\Delta \psi_N, \Delta v) = \lambda_N (\nabla \psi_N, \nabla v), \quad v \in X_N. \quad (4.9)$$

We now give a brief explanation on how to solve the problems (4.9) efficiently. Define the matrices  $A^{m,n}, B^{m,n} \in \mathbb{R}^{N-N_m+1, N-N_n+1}$  with their entries

$$\begin{aligned} A_{j,k}^{m,n} &= 0, \quad |m-n| \notin \{0, 2, 4\}, \\ A_{j,k}^{m,m} &= A_{k,j}^{m,m} = \frac{\pi}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} \right)^2 (\mathcal{L}_m \phi_k^m, \mathcal{L}_m \phi_j^m)_{\omega^{0,1,I}} \\ &\quad + \frac{\pi}{8} \left( \frac{1}{a^2} - \frac{1}{b^2} \right)^2 \left[ (\mathcal{K}_m \phi_k^m, \mathcal{K}_m \phi_j^m)_{\omega^{0,1,I}} + (\mathcal{K}_{-m} \phi_k^m, \mathcal{K}_{-m} \phi_j^m)_{\omega^{0,1,I}} \right], \\ A_{j,k}^{m,m+2} &= A_{k,j}^{m+2,m} = \frac{\pi}{4} \left( \frac{1}{a^4} - \frac{1}{b^4} \right) \left[ (\mathcal{L}_{m+2} \phi_k^{m+2}, \mathcal{K}_m \phi_j^m)_{\omega^{0,1,I}} + (\mathcal{K}_{-m-2} \phi_k^{m+2}, \mathcal{L}_m \phi_j^m)_{\omega^{0,1,I}} \right], \\ A_{j,k}^{m,m+4} &= A_{k,j}^{m+4,m} = \frac{\pi}{8} \left( \frac{1}{a^2} - \frac{1}{b^2} \right)^2 (\mathcal{K}_{-m-4} \phi_k^{m+4}, \mathcal{K}_m \phi_j^m)_{\omega^{0,1,I}}, \end{aligned}$$

and

$$\begin{aligned} B_{j,k}^{m,n} &= 0, \quad |m-n| \notin \{0, 2\}, \\ B_{j,k}^{m,m} &= B_{k,j}^{m,m} = \pi \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \left[ (\partial_r \phi_k^m, \partial_r \phi_j^m)_{\omega^{0,1,I}} + m^2 (\phi_k^m, \phi_j^m)_{\omega^{0,-1,I}} \right], \\ B_{j,k}^{m,m+2} &= B_{k,j}^{m+2,m} = \frac{\pi}{2} \left( \frac{1}{a^2} + \frac{1}{b^2} \right) \left( (\partial_r \phi_k^{m+2} + \frac{m+2}{r} \phi_k^{m+2}, (\partial_r - \frac{m}{r}) \phi_j^m)_{\omega^{0,1,I}} \right). \end{aligned}$$

In view of (3.13)-(3.20), all the nontrivial matrices are penta-diagonal, and their nonzero entries can all be evaluated analytically. Further suppose

$$\psi_N = \sum_{m=-N/2}^{N/2} \sum_{k=N_m}^N \hat{u}_k^m \phi_k^m(r) e^{im\theta}.$$

We arrive at the following algebraic eigenvalue problem:

$$\mathbf{A}\hat{\mathbf{u}} = \lambda_N \mathbf{B}\hat{\mathbf{u}}, \quad (4.10)$$

where  $\hat{\mathbf{u}}$  is the unknown vector

$$\hat{\mathbf{u}} = ((\hat{u}^{-N/2})^t, (\hat{u}^{1-N/2})^t, \dots, (\hat{u}^{N/2})^t)^t, \quad \hat{u}^m = (\hat{u}_{N_m}^m, \hat{u}_{N_m+1}^m, \dots, \hat{u}_N^m)^t,$$

and  $\mathbf{A}$  and  $\mathbf{B}$  are block hepta-diagonal and block penta-diagonal matrices, respectively,

$$\mathbf{A} = [A^{m,n}]_{-N/2 \leq m, n \leq N/2} \quad \text{and} \quad \mathbf{B} = [B^{m,n}]_{-N/2 \leq m, n \leq N/2}.$$

### 4.3 Error estimates

We now conduct the error analysis of (1.6) by using the standard theory of Babuška and J. Osborn [3]. To this end, we first define the semi-norm  $|\cdot|_{s,*}$  in  $H^s(\Omega)$  with  $s \geq 2$ ,

$$|u|_{s,*,\Omega} = \left[ \sum_{\nu=0}^s \|\partial_x^\nu \partial_y^{s-\nu} u\|_{\varpi^{s-2}}^2 + \sum_{\nu=0}^2 \|(a^2 y \partial_x - b^2 x \partial_y)^{s-2} \partial_x^\nu \partial_y^{2-\nu} u\|^2 \right]^{\frac{1}{2}},$$

where the weight function  $\varpi^\alpha = \varpi^\alpha(x, y) = \varpi^\alpha(r) := (1 - r^2)^\alpha$ .

**Theorem 4.1** *Suppose  $u \in H_0^2(\Omega) \cap H^s(\Omega)$  for  $s \geq 2$ . Then it holds that*

$$\inf_{v \in X_N} \|\Delta(u - v)\| \lesssim N^{2-s} |u|_{s,*}. \quad (4.11)$$

*Proof.* We first note that

$$\begin{aligned} \|\Delta u\|^2 &= \int_{\Omega} \left[ |\partial_x^2 u|^2 + (\partial_x^2 u \partial_y^2 \bar{u} + \partial_y^2 u \partial_x^2 \bar{u}) + |\partial_y^2 u|^2 \right] dx dy \\ &= \|\partial_x^2 u\|^2 + 2\|\partial_x \partial_y u\|^2 + \|\partial_y^2 u\|^2, \quad u \in H_0^2(\Omega), \end{aligned}$$

where we derive the second equality sign by integration by parts. Owing to the linear mapping  $(x, y) \mapsto (ax, by)$  from  $D$  onto  $\Omega$ , it suffices to prove (4.11) for  $\Omega$  being the unit disk  $D$ , i.e.,

$$\inf_{v \in X_N} |u - v|_{2,D} \lesssim N^{2-s} \left[ \sum_{\nu=0}^s \|\partial_x^\nu \partial_y^{s-\nu} u\|_{\varpi^{s-2},D}^2 + \sum_{\nu=0}^2 \|(y \partial_x - x \partial_y)^{s-2} \partial_x^\nu \partial_y^{2-\nu} u\|_D^2 \right]. \quad (4.12)$$

To this end, we further denote  $\mathbb{P}_N^{-2}(D) = \mathbb{P}_N(D) \cap H_0^2(D)$  and find that

$$(1 - x^2 - y^2)^2 (x + iy)^m (x - iy)^n = (1 - r^2)^2 r^{m+n} e^{i(m-n)\theta}, \quad m, n \in \mathbb{N}_0, (x, y) \in D.$$

It is then obvious that  $\mathbb{P}_{\min(N, N/2+4)}^{-2}(D) \subset X_{N/2,N} = X_N$  and

$$\inf_{v \in X_N} |u - v|_{2,D} \leq \inf_{v \in \mathbb{P}_{N_*}^{-2}(D)} |u - v|_{2,D}, \quad N_* = \min(N, N/2 + 4).$$

Thus we deduce (4.12) from the following error estimate on polynomial approximations [17, Theorem 4.3],

$$\inf_{v \in \mathbb{P}_N^{-2}} \|u - v\|_{2,D} \lesssim N^{2-s} \left[ \sum_{\nu=0}^s \|\partial_x^\nu \partial_y^{s-\nu} u\|_{\varpi^{s-2},D} + \sum_{\nu=0}^2 \|(y\partial_x - x\partial_y)^{s-2} \partial_x^\nu \partial_y^{2-\nu} u\|_D \right].$$

The proof is now completed. ■

By the approximation theory of Babuška and Osborn on the Ritz method for self-adjoint and positive-definite eigenvalue problems [3, pp. 697-700], we now arrive at the following main theorem.

**Theorem 4.2** *Let  $\{\lambda_{i,N}\}$  be the eigenvalues of (4.9) ordered non-decreasingly with respect to  $i$ , repeated according to their multiplicities. Further let  $\lambda_k$  be an eigenvalue of (1.6) with the geometric multiplicity  $q$  and assume that  $\lambda_k = \lambda_{k+1} = \dots = \lambda_{k+q-1}$ . Then there exists a constant  $C > 0$  such that*

$$\lambda_k \leq \lambda_{j,N} \leq \lambda_k + CN^{4-2s} \sup_{\psi \in E(\lambda_k)} |\psi|_{s,*,\Omega}^2, \quad j = k, k+1, \dots, k+q-1,$$

where

$$E(\lambda_k) := \{\psi \text{ is an eigenfunction corresponding to } \lambda_k \text{ with } \|\psi\|_{1,\Omega} = 1\}.$$

Let  $\psi_{j,N}$  be an eigenfunction corresponding to  $\lambda_{j,N}$  for  $j = k, k+1, \dots, k+q-1$ , then there exists a constant  $C$  such that

$$\inf_{u \in E(\lambda_k)} \|u - \psi_{j,N}\|_{2,\Omega} \leq CN^{2-s} \sup_{\psi \in E(\lambda_k)} |\psi|_{s,*,\Omega}.$$

Let  $\psi_k$  be an eigenfunction corresponding to  $\lambda_k$ , then there exist a constant  $C$  and a function  $v_N \in \text{span}\{\psi_{k,N}, \dots, \psi_{k+q-1,N}\}$  such that

$$\|\psi_k - v_N\|_{2,\Omega} \leq CN^{2-s} \sup_{\psi \in E(\lambda_k)} |\psi|_{s,*,\Omega}.$$

## 5 Numerical experiments

We now perform a sequence of numerical tests to study the convergence behavior and show the effectiveness of our algorithm. We operate our programs in MATLAB 2015b.

### 5.1 Circular disk

#### 5.1.1 Spectral analysis

We now turn to the spectral decomposition of (1.4)–(1.5). Under the polar coordinates, we first reformulate (1.4) as follows,

$$[r^4 \partial_r^2 + 2r^3 \partial_r^3 + (\lambda r^2 + 2\partial_\theta^2 - 1)r^2 \partial_r^2 + (\lambda r^2 - 2\partial_\theta^2 + 1)r \partial_r + (\lambda r^2 + 4 + \partial_\theta^2) \partial_\theta^2] \psi(r, \theta) = 0. \quad (5.1)$$

We next expand  $\psi$  in the Fourier series in  $\theta$ ,

$$\psi(r, \theta) = \sum_{m=-\infty}^{\infty} \psi_m(r) e^{im\theta}.$$

Then (5.1) is reduced to

$$\begin{aligned} [r^4 \partial_r^2 + 2r^3 \partial_r^3 + (\lambda r^2 - 2m^2 - 1)r^2 \partial_r^2 + (\lambda r^2 + 2m^2 + 1)r \partial_r \\ - (\lambda r^2 + 4 - m^2)m^2] \psi_m(r) = 0, \quad \forall m \in \mathbb{Z}, \end{aligned} \quad (5.2)$$

which, together with the pole conditions (2.4)-(2.6), admits a general solution,

$$\psi_m(r) = c_{m,1}(\sqrt{\lambda}r)^m + c_{m,2}J_m(\sqrt{\lambda}r).$$

Meanwhile, the boundary conditions  $\psi_m(1) = \psi'_m(1) = 0$  imply that

$$\begin{cases} c_{m,1}\sqrt{\lambda}^m + c_{m,2}J_m(\sqrt{\lambda}) = 0, \\ c_{m,1}m\sqrt{\lambda}^m + c_{m,2}\sqrt{\lambda}J'_m(\sqrt{\lambda}) = 0, \end{cases} \quad (5.3)$$

with some nontrivial  $c$ 's. As a results, the determinant

$$\lambda^{\frac{m}{2}} [\sqrt{\lambda}J'_m(\sqrt{\lambda}) - mJ_m(\sqrt{\lambda})] = -\lambda^{\frac{m+1}{2}} J_{m+1}(\sqrt{\lambda}) = 0, \quad (5.4)$$

where the second equality sign is derived from the recurrence relation (4) in Page 45 of [28]. In return, the fundamental solution of (5.3) determines the corresponding eigenfunction of (1.4),

$$\psi_m(r, \theta) = [J_m(\sqrt{\lambda_m})(\sqrt{\lambda_m}r)^m - \sqrt{\lambda_m}^m J_m(\sqrt{\lambda_m}r)] e^{im\theta}.$$

Finally, we note that the nontrivial roots of (5.4) define a sequence of increasingly ordered eigenvalues  $0 < \lambda_m^1 < \lambda_m^2 < \lambda_m^3 < \dots$ , which are exactly the eigenvalues of the following second-order equation stemmed from the Laplacian eigenvalue problem on the unit disk,

$$\left( \partial_r^2 + \frac{1}{r} \partial_r - \frac{(1 + |m|)^2}{r^2} \right) \phi_m = \mu_m \phi_m.$$

### 5.1.2 Numerical results

We take  $m = 0, 1, 2$  as our examples. The numerical results of first four eigenvalues for different  $m$  and  $N$  are listed in Table 5.1-5.4.

**Table 5.1 The first four eigenvalues for  $m = 0$  and different  $N$  in the unit disk.**

N	$\lambda_{0N}^1$	$\lambda_{0N}^2$	$\lambda_{0N}^3$	$\lambda_{0N}^4$
10	14.6819706421365	49.2184567483993	103.5024835613828	177.6009453441972
20	14.6819706421239	49.2184563216945	103.4994538951366	177.5207668138042
30	14.6819706421239	49.2184563216945	103.4994538951365	177.5207668138044

**Table 5.2 The first four eigenvalues for  $m = 1$  and different  $N$  in the unit disk.**

N	$\lambda_{1N}^1$	$\lambda_{1N}^2$	$\lambda_{1N}^3$	$\lambda_{1N}^4$
20	26.3746164271634	70.8499989190960	135.0207088659703	218.9201891456649
30	26.3746164271634	70.8499989190958	135.0207088659704	218.9201891456624
40	26.3746164271634	70.8499989190957	135.0207088659696	218.9201891456631
50	26.3746164271634	70.8499989190957	135.0207088659700	218.9201891456630

**Table 5.3 The first four eigenvalues for  $m = 2$  and different  $N$  in the unit disk.**

N	$\lambda_{2N}^1$	$\lambda_{2N}^2$	$\lambda_{2N}^3$	$\lambda_{2N}^4$
30	40.7064658182003	95.2775725440372	169.3954498260997	263.2008542550081
40	40.7064658182002	95.2775725440371	169.3954498260988	263.2008542550071
50	40.7064658182004	95.2775725440372	169.3954498260995	263.2008542550078
60	40.7064658182003	95.2775725440370	169.3954498260994	263.2008542550076

We know from Tables 5.1–5.3 that numerical eigenvalues achieve at least fourteen-digit accuracy with  $N \geq 20$  for  $m = 0$  and  $N \geq 40$  for  $m = 1, 2$ , respectively. If we choose the numerical solutions with  $N = 60$  as reference solutions, the error figures of the approximate eigenvalue  $\lambda_{mN}^i$  ( $m = 0, 1, 2; i = 1, 2, 3, 4$ ) with different  $N$  are listed in Figures 1-3.

It is worthy to note that, when imposing the pole conditions  $\psi'_m(0) = \psi_m(0) = 0$  as in [23] for  $m = 1$ , one would necessarily get spurious eigenvalues even for large  $N$ , which can only serve as upper bounds of each the exact ones. For instance, the first computational eigenvalue in this case reads 28.7378, a number far away from the reference one 26.3746.

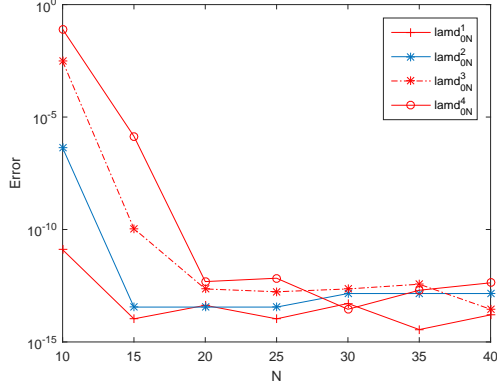


Figure 1: Errors between numerical solutions and the reference solution for  $m = 0$ .

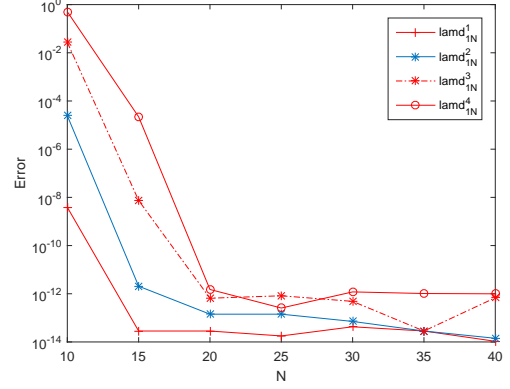


Figure 2: Errors between numerical solutions and the reference solution for  $m = 1$ .

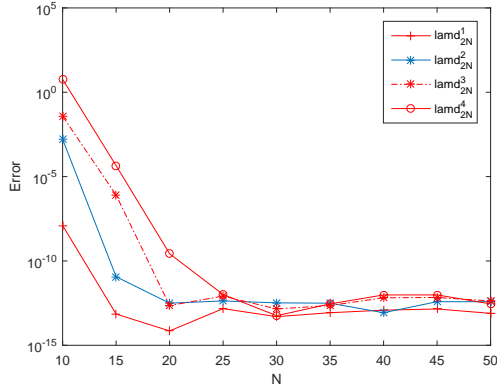


Figure 3: Errors between numerical solutions and the reference solution for  $m = 2$  on the unit disk.

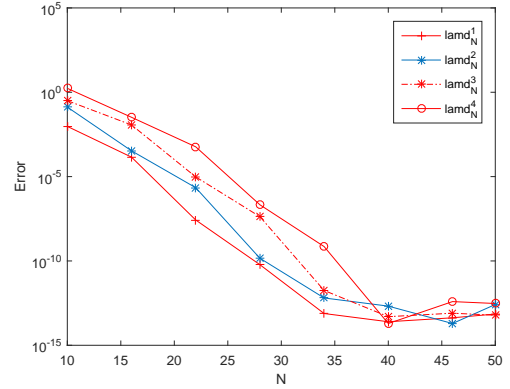


Figure 4: Errors between numerical solutions and the reference solution on the elliptic domain.

## 5.2 Elliptic domain

We take  $a = 3, b = 1$  as our example. The numerical data of the first four eigenvalues are listed in Table 5.4. We see that the eigenvalues achieve at least fourteen-digit accuracy with  $N \geq 40$ . If we choose the solutions of  $N = 60$  as reference solutions, the error figures of the approximate eigenvalue  $\lambda_N^i$  ( $i = 1, 2, 3, 4$ ) with different  $N$  are listed in Figure 4.

**Table 5.4 The first four eigenvalues for different  $N$  in an elliptic domain with  $a = 3, b = 1$ .**

$N$	$\lambda_N^1$	$\lambda_N^2$	$\lambda_N^3$	$\lambda_N^4$
20	9.96633619654313	11.0706597920227	13.1630821009849	15.6448857440637
30	9.9663343484475	11.0706554383893	13.1627539459867	15.6437495616386
40	9.96633434844728	11.0706554383168	13.1627539455290	15.6437494538630
50	9.96633434844729	11.0706554383167	13.1627539455291	15.6437494538630
60	9.96633434844726	11.0706554383166	13.1627539455290	15.6437494538630

Before concluding this section, we would like to present some figures of the (real) eigenfunctions corresponding to the smallest 8 eigenvalues in Figures 5-12.

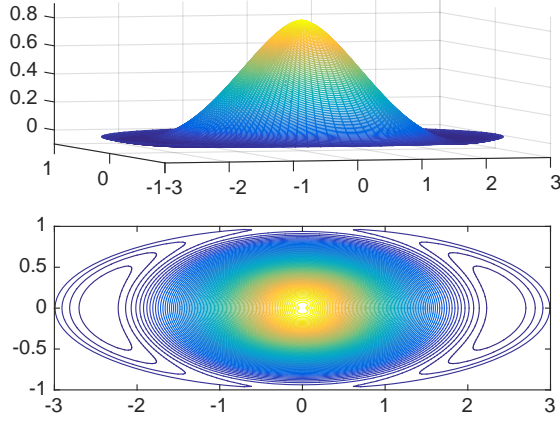


Figure 5: Mesh and contour of  $\psi_1$ .

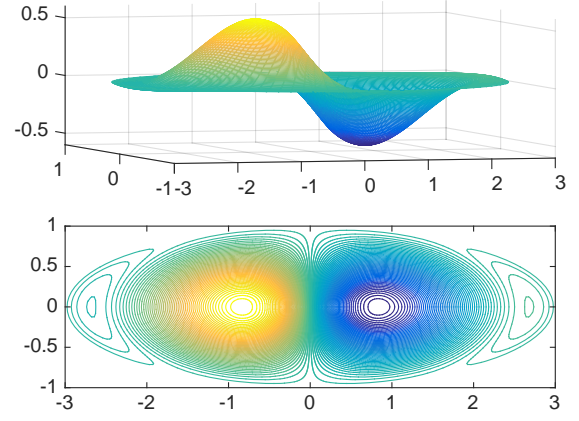


Figure 6: Mesh and contour of  $\psi_2$ .

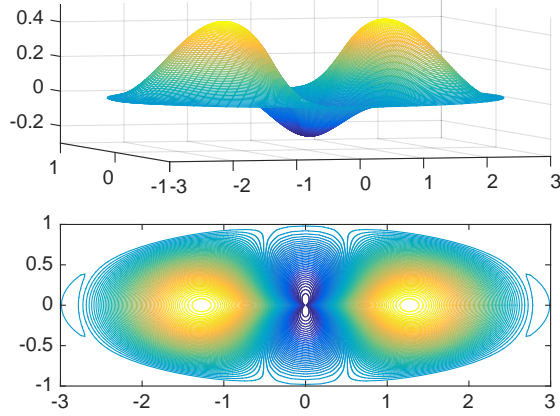


Figure 7: Mesh and contour of  $\psi_3$ .

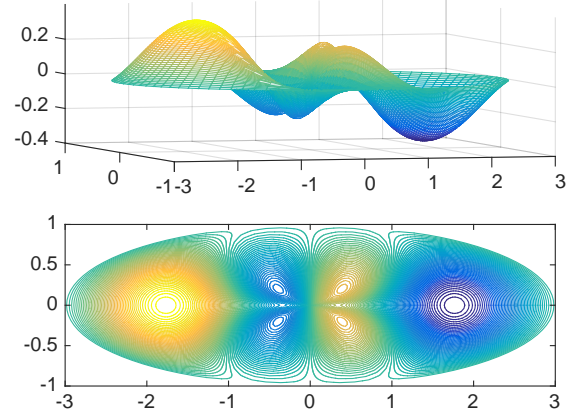


Figure 8: Mesh and contour of  $\psi_4$ .

## 6 Conclusions

We present a rigorous error analysis for our proposed spectral-Galerkin methods in solving the Stokes eigenvalue problem under the stream function formulation in polar geometries. We derive the essential pole condition and reduce the problem to a sequence of one-dimensional eigenvalue problems that can be solved individually in parallel. Spectral accuracy is achieved by properly designed non-polynomial basis functions and the exponential rate of convergence is established by introducing a suitable weighted Sobolev space; all based on the correct pole condition. To the best of our knowledge, the pole condition and such kind of usage of weighted Sobolev space and



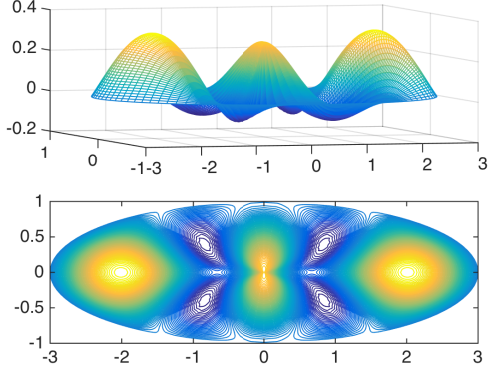


Figure 9: Mesh and contour of  $\psi_5$ .

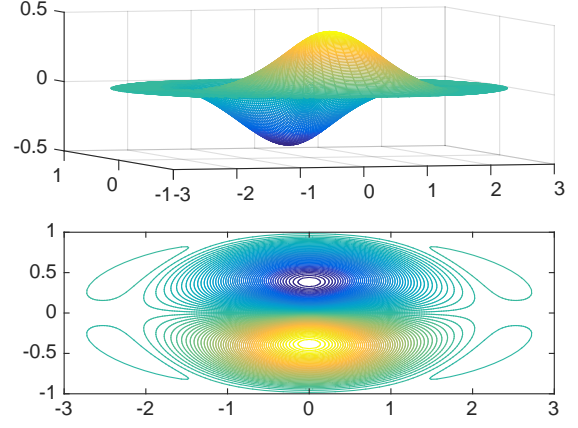


Figure 10: Mesh and contour of  $\psi_6$ .

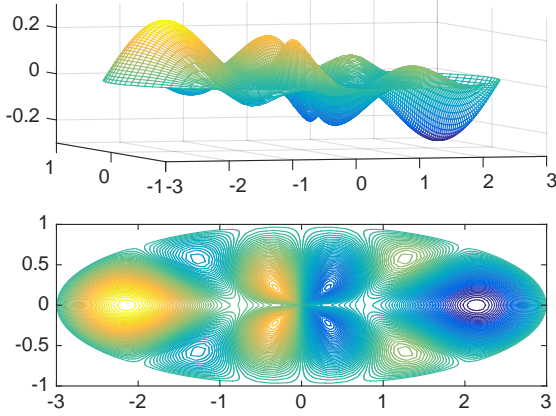


Figure 11: Mesh and contour of  $\psi_7$ .

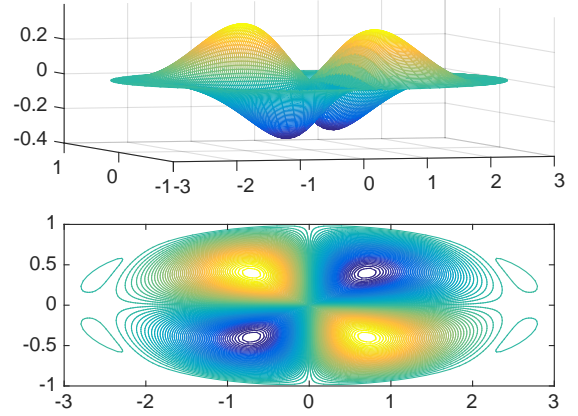


Figure 12: Mesh and contour of  $\psi_8$ .

basis functions are all for the first time in the literature. Our spectral-Galerkin method is also extended to solve the stream function formulation of the Stokes eigenvalue problem on an elliptic region, which also indicates the capability of our method to solve fourth-order equations on other smooth domains. Numerical experiments in the last section have validated the theoretical results and algorithms. As we can see, on special domains such as circular disks and elliptic regions, with only less than 50 degrees of unknowns, the proposed spectral method can achieve 14-digits accuracy for the first few eigenvalues of the Stokes problem, this is far more superior to traditional methods such as finite element and finite difference methods.

## A Jacobi and generalized Jacobi polynomials

The classical Jacobi polynomials  $J_k^{\alpha,\beta}(\zeta)$ ,  $k \geq 0$  with  $\alpha, \beta > -1$  are mutually orthogonal with respect to the Jacobi weight function  $\chi^{\alpha,\beta} := \chi^{\alpha,\beta}(\zeta) = (1 - \zeta)^\alpha(1 + \zeta)^\beta$  on  $\Lambda = (-1, 1)$ ,

$$\int_{-1}^1 J_m^{\alpha,\beta}(\zeta) J_n^{\alpha,\beta}(\zeta) \chi^{\alpha,\beta}(\zeta) d\zeta = \frac{2^{\alpha+\beta+1}}{2n + \alpha + \beta + 1} h_n^{\alpha,\beta} \delta_{m,n}, \quad m, n \geq 0, \quad (\text{A.1})$$

where  $\delta_{m,n}$  is the Kronecker delta, and

$$h_n^{\alpha,\beta} := \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{\Gamma(n + 1)\Gamma(n + \alpha + \beta + 1)}. \quad (\text{A.2})$$

For  $k \in \mathbb{Z}$ , denote by  $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$  the Pochhammer symbol. The classical Jacobi polynomials possess the following important representation

$$J_n^{\alpha,\beta}(\zeta) = \sum_{k=0}^n \frac{(-n-\beta)_{n-k}(n+\alpha+\beta+1)_k}{(n-k)!k!} \left(\frac{\zeta+1}{2}\right)^k, \quad (\text{A.3})$$

which symbolically furnishes the extension of  $J_n^{\alpha,\beta}(\zeta)$  to arbitrary  $\alpha$  and  $\beta$ . Generalized Jacobi polynomials preserve most of the essential properties of the classic Jacobi polynomials, among which the following identities [27] are of importance in the current paper,

$$J_n^{\alpha,\beta}(-\zeta) = (-1)^n J_n^{\beta,\alpha}(\zeta), \quad (\text{A.4})$$

$$\partial_\zeta J_n^{\alpha,\beta}(\zeta) = \frac{n+\alpha+\beta+1}{2} J_{n-1}^{\alpha+1,\beta+1}(\zeta), \quad (\text{A.5})$$

$$J_n^{\alpha,\beta}(\zeta) = \sum_{\nu=n-k}^n \frac{2\nu+\alpha+\beta+k+1}{(n+\nu+\alpha+\beta+1)_{k+1}} \frac{(\nu+\beta+1)_{n-\nu}}{(\nu+\alpha+\beta+k+1)_{n-\nu-k}} \frac{(-k)_{n-\nu}}{(n-\nu)!} J_\nu^{\alpha+k,\beta}(\zeta), \quad (\text{A.6})$$

$$(1+\zeta)J_n^{\alpha,\beta+1}(\zeta) = \frac{2(n+\beta+1)}{2n+\alpha+\beta+2} J_n^{\alpha,\beta}(\zeta) + \frac{2(n+1)}{2n+\alpha+\beta+2} J_{n+1}^{\alpha,\beta}(\zeta). \quad (\text{A.7})$$

In particular, the generalized Jacobi polynomials with  $\alpha$  and/or  $\beta$  being integers are our greatest interest [16],

$$J_n^{\alpha,\beta}(\zeta) = \begin{cases} \left(\frac{\zeta-1}{2}\right)^{-\alpha} \left(\frac{\zeta+1}{2}\right)^{-\beta} J_{n+\alpha+\beta}^{-\alpha,-\beta}(\zeta), & \alpha, \beta \in \mathbb{Z}, n+\alpha+\beta \in \mathbb{N}_0, \\ h_n^{\alpha,\beta} \left(\frac{\zeta-1}{2}\right)^{-\alpha} J_{n+\alpha}^{-\alpha,\beta}(\zeta), & \alpha \in \mathbb{Z}, n+\alpha \in \mathbb{N}_0, \\ h_n^{\alpha,\beta} \left(\frac{\zeta+1}{2}\right)^{-\beta} J_{n+\beta}^{\alpha,-\beta}(\zeta), & \beta \in \mathbb{Z}, n+\beta \in \mathbb{N}_0. \end{cases} \quad (\text{A.8})$$

The generalized Jacobi polynomials with negative indices not only simplify the numerical analysis for the spectral approximations of differential equations, but also lead to very efficient numerical algorithms [12, 24].

Finally, it is worthy to point out that a reduction of the degree of  $J_n^{\alpha,\beta}(\zeta)$  occurs if and only if  $-n-\alpha-\beta \in \{1, 2, \dots, n\}$ ,

$$J_n^{\alpha,\beta}(\zeta) = h_n^{\alpha,n_0-n-1} J_{n_0-1}^{\alpha,\beta}(\zeta), \quad (\text{A.9})$$

where  $n_0 := -n-\alpha-\beta$  if  $-n-\alpha-\beta \in \{1, 2, \dots, n\}$  and  $n_0 := 0$  otherwise.

## B Proof of Lemma 3.5

At first, (3.18)-(3.20) are trivial results on the Jacobi expansion.

By (A.4), (A.6), (A.8), one finds that, for  $i \geq 4$ ,

$$(1-t^2)^2 J_{i-4}^{2,1}(t) = \frac{1}{2}(1-t^2)^2 \left[ \frac{i}{i-2} J_{i-4}^{2,2}(t) + J_{i-5}^{2,2}(t) \right] = \frac{8i}{i-2} J_i^{-2,-2}(t) + 8(1-\delta_{i,4}) J_{i-1}^{-2,-2}(t).$$

Then by (A.5), (A.4) and (A.6), one derives

$$\begin{aligned} \partial_t^2 [(1-t^2)^2 J_{i-4}^{2,1}(t)] &= 2i(i-3) J_{i-2}^{0,0}(t) + 2(i-4)(i-3) J_{i-3}^{0,0}(t) \\ &= \frac{2i(i-3)}{2i-3} ((i-1) J_{i-2}^{0,1}(t) + (i-2) J_{i-3}^{0,1}(t)) + \frac{2(i-4)(i-3)}{2i-5} ((i-2) J_{i-3}^{0,1}(t) + (i-3) J_{i-4}^{0,1}(t)) \\ &= \frac{2(i-3)(i-1)i}{(2i-3)} J_{i-2}^{0,1}(t) + \frac{8(i-3)^2(i-2)(i-1)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(t) + \frac{2(i-4)(i-3)^2}{(2i-5)} J_{i-4}^{0,1}(t), \end{aligned}$$

and

$$\begin{aligned}
\frac{1}{t+1} \partial_t [(1-t^2)^2 J_{i-4}^{2,1}(t)] &= \frac{1}{t+1} \left[ \frac{4i(i-3)}{i-2} J_{i-1}^{-1,-1}(t) + 4(i-4) J_{i-2}^{-1,-1}(t) \right] \\
&= \left[ \frac{2i(i-3)}{i-2} J_{i-2}^{-1,1}(t) + \frac{2(i-4)(i-3)}{i-2} J_{i-3}^{-1,1}(t) \right] \\
&= \frac{2(i-3)i}{(2i-3)} J_{i-2}^{0,1}(t) - \frac{12(i-3)(i-2)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(t) - \frac{2(i-4)(i-3)}{(2i-5)} J_{i-4}^{0,1}(t),
\end{aligned}$$

which give (3.13) and (3.14) immediately.

Next by (A.8) and (A.6),

$$\begin{aligned}
\frac{1}{(t+1)^2} (1-t^2)^2 J_{i-4}^{2,1}(t) &= (1-t)^2 J_{i-4}^{2,1}(t) = \frac{4(i-3)}{i-1} J_{i-2}^{-2,1}(t) \\
&= \frac{2(i-3)}{2i-3} J_{i-2}^{0,1}(t) - \frac{8(i-3)(i-2)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(t) + \frac{2(i-3)}{2i-5} J_{i-4}^{0,1}(t),
\end{aligned}$$

which states (3.16).

Further, by (A.5), (A.4) and (A.6),

$$\begin{aligned}
\partial_t [(1-t^2)^2 J_{i-4}^{2,1}(t)] &= \frac{4i(i-3)}{i-2} J_{i-1}^{-1,-1}(t) + 4(i-4) J_{i-2}^{-1,-1}(t) \\
&= \frac{2(i-3)i^2}{(2i-3)(2i-1)} J_{i-1}^{0,1}(t) + \frac{4(i-1)(i-3)(2i^2-7i+2)}{(2i-5)(2i-1)(2i-3)} J_{i-2}^{0,1}(t) \\
&\quad - \frac{8(i-2)(i-3)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(t) - \frac{4(2i^2-9i+6)(i-3)^2}{(2i-7)(2i-5)(2i-3)} J_{i-4}^{0,1}(t) - \frac{2(i-3)(i-4)^2}{(2i-5)(2i-7)} J_{i-5}^{0,1}(t),
\end{aligned}$$

and by (A.8), (A.4) and (A.6),

$$\begin{aligned}
(1-t)^2 (1+t) J_{i-4}^{2,1}(t) &= 8 J_{i-1}^{-2,-1}(t) = \frac{2i(i-3)}{(2i-1)(2i-3)} J_{i-1}^{0,1}(t) - \frac{8(i-1)(i-3)}{(2i-1)(2i-3)(2i-5)} J_{i-2}^{0,1}(t) \\
&\quad - \frac{4(i-2)(i-3)}{(2i-3)(2i-5)} J_{i-3}^{0,1}(t) + \frac{8(i-3)^2}{(2i-7)(2i-3)(2i-5)} J_{i-4}^{0,1}(t) + \frac{2(i-3)(i-4)}{(2i-5)(2i-7)} J_{i-5}^{0,1}(t),
\end{aligned}$$

which lead to (3.15) and (3.17), respectively.

Finally, (3.21) and (3.22) are direct consequences of (3.13)-(3.20) and (A.1). ■

## References

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